Diffrencial Equations

* Differential Equations *

Eqn - $x+y=2$ diff. eqn - $\frac{dy}{dx} x+y=2$

soln $\Rightarrow x=1$ & $y=1$ soln $\Rightarrow y=f(x)$

Differential eqn

ordinary diff. eqn — single variable

$$\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^3+3y=0$$

partial diff. eqn — more than one var

$$\frac{\partial^2 u}{\partial x^2}+\left(\frac{\partial^2 u}{\partial y^2}\right)^3=0$$

order = 2
degree = 3

order = highest derivative
degree = highest power of highest degree derivative should free from radicals & fractions.

order = 2 ; degree = 1

order - The highest derivative involved in diff. eqn.

Radicals = $\left(\frac{dy}{dx}\right)^{1/2, 1/3, 1/4 \ldots}$

fraction - derivative should not in denomenator

① $\left(\frac{d^2y}{dx^2}\right)^2 = \left(x+\frac{dy}{dx}\right)^{3/2}$

$\left(\frac{d^2y}{dx^2}\right)^4 = \left(x+\frac{dy}{dx}\right)^3$

$\therefore O=2 \quad D=4$

② $\frac{dy}{dx} = x + \frac{e^y}{\frac{dy}{dx}}$

$\left(\frac{dy}{dx}\right)^2 = x\frac{dy}{dx}+e^y$

$O=1, D=2$

* Differential Equations *

Eqⁿ — $x+y=2$ diff eqn $\frac{dy}{dx} x + y = 2$

solⁿ ⇒ $x=1$ & $y=1$ solⁿ ⇒ $y = f(x)$

Differential eqn

↓ Ordinary diff. eqn

↓ partial diff. eqn

single variable

$$\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 + 3y = 0$$

more than one var

$$\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial^2 u}{\partial y^2}\right)^3 = 0$$

order = 2

degree = 3

order = highest derivative

degree = highest power of highest degree derivative should free from radicals & fractions.

order = 2 ; degree = 1

order – The highest derivative involved in diff. eqⁿ.

Radical – $\left(\frac{dy}{dx}\right)^{1/2, 1/3, 1/4 \dots}$

fraction – derivative should not in denomenator

1) $\left(\frac{d^2y}{dx^2}\right)^2 = \left(x + \frac{dy}{dx}\right)^{3/2}$

$\left(\frac{d^2y}{dx^2}\right)^4 = \left(x + \frac{dy}{dx}\right)^3$

∴ O = 2 D = 4

2) $\frac{dy}{dx} = x + \frac{e^y}{\frac{dy}{dx}}$

$\left(\frac{dy}{dx}\right)^2 = x\frac{dy}{dx} + e^y$

O = 1, D = 2

) $\left(\frac{d^3y}{dx^3}\right)^2 + \left(\frac{d^2y}{dx^2}\right)^3 + 3y = 0$

$O = 3$, $D = 2$

④ $dy = (t + \sin t)\, dt$

$\frac{dy}{dt} = t + \sin t$

$O = 1$, $D = 1$

) $\left(\frac{\partial^2 u}{\partial x^2}\right)^3 = c^2 \left(\frac{\partial^3 y}{\partial y^3}\right)^2$

$O = 3$, $D = 2$

Formation of differential eqns -

$F(x, y, a, b) = 0$ — ①

Diff. ① w.r.t. x

$F_1\left(x, y, \frac{dy}{dx}, a, b\right) = 0$ — ②

Diff. ② w.r.t. x

$F_2\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, a, b\right) = 0$ — ③

Eliminate $a, b \Rightarrow \phi\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right) = 0$

Note:

The no. of arbitrary constants eliminated should be equal to the order of resulting ordinary diff. eqns.

1. The diff eqn of family of straight lines passing through origin is

(a) $ydy - xdx = 0$ (b) $xdy - ydx = 0$

(c) $xdx + ydy = 0$ (d) none

→ $y = mx$

$\frac{dy}{dx} = m$

$\therefore \quad y = \frac{dy}{dx} x$

$dy\,x = dx \cdot y$

$\therefore \ xdy - ydx = 0$

2. Find the diff. eqn of family of circle centred at (h, k) & radius 'a' where h, k are arbitrary constants.

$$(x-h)^2 + (y-k)^2 = a^2 \quad \text{—— (1)}$$

diff w.r.t. x

$$2(x-h) + 2(y-k)y' = 0 \qquad x-h = -(y-k)y' \quad \text{—— (2)}$$

diff w.r.t. x

$1 = -y'y' - (y-k)y''$ [illegible]

[illegible] $\frac{1+(y')^2}{y''}$ [illegible] $\frac{(\ldots)}{y''}$ [illegible]

$$x-h = \frac{(1+y'^2)\cdot y'}{y''} \quad \text{--- (4)}$$

Put (4) in (1)

$$\frac{(1+y'^2)^2}{y''^2}y'^2 + \frac{(1+y'^2)^2}{y''^2} = a^2$$

$$(1+y'^2)^2(y'^2+1) = a^2 y''^2$$

$$(1+y'^2)^3 = a^2 y''^2$$

(3) $y = c(x-c)^2 \quad \text{--- (1)}$

Diff. w.r.t. x

$$y' = 2c(x-c) \quad \leftarrow (2)$$

$$\frac{(1)}{(2)} \Rightarrow \frac{y}{y'} = \frac{\cancel{c}(x-c)^{\cancel{2}}}{2\cancel{c}\cancel{(x-c)}} = \frac{x-c}{2} \quad \therefore x-c = \frac{2y}{y'}$$

$$c = x - \frac{2y}{y'} = \frac{xy'-2y}{y'}$$

$$y = \frac{xy'-2y}{y'}\left(x - \frac{xy'-2y}{y'}\right)^2 = \left(\frac{xy'-2y}{y'}\right)\frac{4y^2}{y'^2}$$

$$y'^3 = (xy'-2y)4y$$

Note :- If the given eqn is of the form $y = c_1 e^{ax} + c_2 e^{bx} + c_3 e^{cx} + \cdots$

Then D.E is

$(D-a)(D-b)(D-c)y = 0$

$D = \frac{d}{dx}$; $D^2 = \frac{d^2}{dx^2}$,

e.g.

① $y = c_1 e^{-x} + c_2 e^{3x} + c_3 e^{2x}$

$\rightarrow$

$(D+1)(D-3)(D-2)y = 0$

$(D^2 - 2D - 3)(D-2)y = 0$

$(D^3 - 4D^2 + D + 6)y = 0$

$\frac{d^3y}{dx^3} - 4\frac{d^2y}{dx^2} + \frac{dy}{dx} + 6y = 0$

② $i = c_1 e^{2t} + c_2 e^{-4t}$

$\rightarrow$ $(D-2)(D+4)i = 0$

$(D^2 + 2D - 8)i = 0$

$\frac{d^2i}{dt^2} + 2\frac{di}{dt} - 8i = 0$

Note:-

If the given eqn is of the form

$y = A f(x) + B g(x)$ Then the D.E. is obtained by simplifying the following determinant.

$$\begin{vmatrix} y & f(x) & g(x) \\ y' & f' & g' \\ y'' & f'' & g'' \end{vmatrix} = 0$$

e.g. ① $y = A e^{2x} + Bx$

$$\begin{vmatrix} y & e^{2x} & x \\ y' & 2e^{2x} & 1 \\ y'' & 4e^{2x} & 0 \end{vmatrix} = 0$$

$\therefore y(-4e^{2x}) - e^{2x}(-y'') + x(4y'e^{2x} - 4y''e^{2x}) = 0$

$\therefore y''(1-2x) + 4xy' - 4y = 0.$

② $y = e^{x}(A\cos x + B\sin x)$ — ①

$y' = e^{x}(A\cos x + B\sin x) + e^{x}(-A\sin x + B\cos x)$ — ②

$y' = y + e^{x}(-A\sin x + B\cos x)$

$y'' = y' + e^{x}(-A\sin x + B\cos x) + e^{x}(-A\cos x - B\sin x)$

$y'' = y' + (y' - y) - y$

$\therefore y'' - 2y' + 2y = 0$

$$\boxed{\frac{d^2y}{dt^2} = g}$$

$$\boxed{y(0) = 0, \quad y'(0) = 0}$$

$\frac{dy}{dt} = gt + c_1$ — $y'=0$, $t=0$: $0 = 0 + c_1 \therefore c_1 = 0$

$y = \frac{gt^2}{2} + c_1 t + c_2$ — $y=0$, $t=0$: $0 = 0 + c_2 \therefore c_2 = 0$

General sol^n — 2: no. of arbitrary ~~cnst~~

$\boxed{y = \frac{gt^2}{2}}$ P. sol^n

$$\boxed{\frac{d^2y}{dt^2} = g}$$

(a) $y = \frac{1}{2}gt^2 + c_1 e^t + c_2$

(b) $y = \frac{1}{2}gt^2 + c_1 t^3 + c_2$

(c) $y = \frac{1}{2}gt^2 + c_1 t + c$ ✓

(d) None

First order first degree diff. eq^{ns}

Notation - $\boxed{\frac{dy}{dx} = F(x,y)}$ or $\boxed{M\,dx + N\,dy = 0}$ $M(x,y)$ & $N(x,y)$

① Variable separable method -

$$\frac{dy}{dx} = F(x,y)$$

$$f(x)\,dx = g(y)\,dy$$

$$\int f(x)\,dx = \int g(y)\,dy + c$$

① $\log\left(\frac{dy}{dx}\right) = 2x-y$

$\rightarrow$

$\cancel{\log dy - \log dx = 2x-y}$

$\frac{dy}{dx} = e^{2x-y} = e^{2x} \times e^{-y}$

$\int e^{y}dy = \int e^{2x}dx$

$$\boxed{e^{y} = \frac{e^{2x}}{2} + c}$$

② $\frac{dy}{dx} = \frac{y^2}{1-xy}$

$(1-xy)\,dy = y^2\,dx$

$dy = y^2dx + xy\,dy$

$= y\,(\underbrace{y\,dx + x\,dy})$

$dy = y\,d(xy)$

$\int \frac{dy}{y} = \int d(xy)$

$$\boxed{\log y = xy + c}$$

③ $\frac{dx}{dy} = e^{x-2y} + e^{x}.y$

$\frac{dx}{dy} = e^{x}(e^{-2y}+y)$

$\int \frac{dx}{e^{x}} = \int (e^{-2y}+y)\,dy$

$-e^{-x} = \frac{e^{-2y}}{-2} + \frac{y^2}{2} + c$

④ $\frac{dy}{dx} = 1+y^2$

$\int \frac{dy}{1+y^2} = \int dx$

$\tan^{-1}y = x+c$

$$\boxed{y = \tan(x+c)}$$

⑤ $\frac{dy}{dx} = -\frac{x}{y}$ at $x=1, y=\sqrt{3}$

$\int y\,dy = \int -x\,dx$

$y^2 = -x^2 + 2c$

$3 = -1 = 2c$

$2c = 4$

$\boxed{c=2}$

$$\boxed{x^2+y^2=4}$$

⑥ Find the curve passing through the point (0,1) and satisfing $\sin\left(\frac{dy}{dx}\right) = b$

→

$$\sin\left(\frac{dy}{dx}\right) = b$$

$$\frac{dy}{dx} = \sin^{-1} b$$

$$\int dy = \int \sin^{-1} b \, dx$$

$$y = x \sin^{-1} b + c$$

$$\boxed{c = 1}$$

$$\boxed{y = x \sin^{-1} b + 1}$$

⑦ $\frac{dx}{dt} + 3x = 0$

ⓐ $x = 3e^{3t}$ ⓑ $x = 3t^3$

ⓒ $x = 2e^{-2t}$ ✓ⓓ $x = 3e^{-3t}$

→

$$\frac{dx}{dt} = -3x$$

$$\frac{dx}{x} = -3dt$$

$$\log x = -3t + \log c$$

$$x = ce^{-3t}$$

$$\therefore \boxed{x = 3e^{-3t}}$$

⑧ $\frac{dy}{dx} = e^{x+y}$ with $y(1) = 1$

then find 'y' when $x = -1$

→

$$\int e^{-y} dy = \int e^{x} dx$$

$$-e^{-y} = e^{x} + c$$

$$-e^{-1} = e^{1} + c$$

$$-\frac{1}{e} - e = c$$

$$\therefore c = \frac{-1 - e^2}{e}$$

$$-e^{-y} = e^{x} - e^{-1} - e$$

$x = -1$

$$-e^{-y} = \cancel{e^{-1}} - \cancel{e^{-1}} - e$$

$$e^{-y} = e$$

$$\therefore -y = 1$$

$$\boxed{y = -1}$$

⑨ $\frac{dy}{dx} = 3x^2 - 2x$ at $x=1, y=1$ then find $y(3) = ?$

→ $\int dy = \int (3x^2 - 2x)\, dx$

$y = x^3 - x^2 + c$

$\boxed{c=1}$

$y = x^3 - x^2 + 1$

$y(3) = 27 - 9 + 1$

$\boxed{y(3) = 19}$

⑩ $\frac{dy}{dx} = y^2 \sin x$ with $y(\pi) = 1$

ⓐ $y \sin x = 1$ ✓ⓑ $y \cos x = 1$

ⓒ $y = \sin x$ ⓓ $x = \sin y$

⑪ If the equation contains the terms like $\cos(xy)$, $\sin(x+y)$, $(ax+by+c)^2$, etc. can be reduced to variable separable form with a substitution $xy = v$, $x+y = v$, $ax+by+c = v$ resp.

① $\frac{dy}{dx} = (4x + y + 1)^2$

$4x + y + 1 = v$

$4 + \frac{dy}{dx} = \frac{dv}{dx}$

$\frac{dy}{dx} = \frac{dv}{dx} - 4$

$\therefore \frac{dv}{dx} - 4 = v^2$

$\int \frac{dv}{(4 + v^2)} = \int dx$

$\frac{1}{2} \tan^{-1} \frac{v}{2} = x + c$

$\tan^{-1} \frac{v}{2} = 2x + 2c$ $\therefore \boxed{\tan(2x+2c) = \frac{4x+y+1}{2}}$

Homogeneous diff method -

$$\frac{dy}{dx} = F(x,y)$$

~~Hom~~ f^n of degree n

Num. degree - den. degree

$$F(kx, ky) = k^n F(x,y)$$

$\frac{dy}{dx} = F(x,y)$ is said to be homogeneous diff eqn if $F(x,y)$ should be a homogeneous function of degree zero.

Note - Eq^n $Mdx + Ndy = 0$ is said to homogeneous diff. eqn if all the terms of M & N should be of same degree.

$$x^2+y^2 \Rightarrow (kx)^2 + (ky)^2 \quad \therefore \text{ degree} = 2.$$

Substitution $\boxed{y = vx}$ or $\boxed{x = vy}$ reduces homogeneous eqn to variable separation form.

$$\frac{dy}{dx} = \frac{x^2y - y^3}{xy^2} = \frac{\cancel{x^3}\left(y/x - (y/x)^3\right)}{\cancel{x^3}(y/x)^2}$$

Every homogeneous f^n of degree zero can be written as f^n of y/x or x/y & substitution reduces it to variable separation form.

$$x\frac{dy}{dx} = y\{\log y - \log x + 1\}$$

$$\frac{dy}{dx} = \frac{y}{x}\left[\log\left(\frac{y}{x}\right) + 1\right]$$

$$y = vx$$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$v + x\frac{dv}{dx} = v\left[\log v + 1\right]$$

$$x\frac{dv}{dx} = v\log v$$

$$\int \frac{1}{v\log v}\,dv = \int \frac{dx}{x}$$

$$\log v = t$$

$$\frac{1}{v}\,dv = dt$$

$$\int \frac{1}{t}\,dt = \int \frac{dx}{x}$$

$$\log t = \log x + \log c$$

$$t = cx$$

$$\log v = cx$$

$$\therefore v = e^{cx}$$

$$\therefore \frac{y}{x} = e^{cx}$$

$$\therefore \boxed{y = xe^{cx}}$$

Non-homogeneous diff. eqn method -

$$\boxed{\frac{dy}{dx} = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}}$$

case I :- $\left[\frac{a_1}{a_2} = \frac{b_1}{b_2}\right]$

There exists a subst. which reduces given eqn to variable separable form.

case II :- $\left[\frac{a_1}{a_2} \neq \frac{b_1}{b_2}\right]$

$x = X+h \quad , \quad y = Y+K$

$dx = dX \quad , \quad dy = dY$

$$\frac{dY}{dX} = \frac{a_1(X+h)+b_1(Y+K)+c_1}{a_2(X+h)+b_2(Y+K)+c_2}$$

$$= \frac{a_1X+b_1Y+(a_1h+b_1K+c_1)}{a_2X+b_2Y+(a_2h+b_2K+c_2)}$$

choose h, K so that

$$\left.\begin{array}{l} a_1h+b_1K+c_1=0 \\ a_2h+b_2K+c_2=0 \end{array}\right\}$$

$$\frac{dY}{dX} = \frac{a_1X+b_1Y}{a_2X+b_2Y}$$ then $Y = VX$

(1) $(2x+2y-1)dx = (x+y+1)\,dy$

$$\frac{dy}{dx} = \frac{2x+2y-1}{x+y+1} \qquad \left(\frac{a_1}{a_2} = \frac{b_1}{b_2}\right)$$

$$= \frac{2(x+y)-1}{(x+y)+1}$$

$x+y = v$

$$1 + \frac{dy}{dx} = \frac{dv}{dx}$$

$$\frac{dy}{dx} = \frac{dv}{dx} - 1$$

$$\frac{dv}{dx} - 1 = \frac{2v-1}{v+1}$$

$$\frac{dv}{dx} = \frac{2v-1+v+1}{v+1} = \frac{3v}{v+1}$$

$$\int \frac{(v+1)}{v}\,dv = \int 3\,dx$$

$$v + \log v = 3x + c$$

$$x+y + \log(x+y) = 3x + c$$

$$y - 2x + \log(x+y) = c$$

17

(7) Which of following subst. reduces diff. eqn $\frac{dy}{dx} = \frac{y+x-2}{y-x-4}$ to homogeneous form.

$\rightarrow$

$x = X + h$, $y = Y + k$

$k + \cancel{h} - 2 = 0$
$k - \cancel{h} - 4 = 0$

$2k - 6 = 0$

$k = 3$, $h = -1$

$\boxed{x = X - 1}$, $\boxed{y = Y + 3}$

$\frac{dY}{dX} = \frac{Y+X}{Y-X}$ Homo. form

Linked que.

(8) (a) $x = X + h$, $y = Y + k$ reduces $\frac{dy}{dx} = \frac{y+x-2}{y-x-4}$ to homo. form then find h, k.

Ans. $h = -1$, $k = 3$

(b) $\frac{d^2y}{dx^2} + (h+3)\frac{dy}{dx} + (k-1)y = 0$

* Exact differential eqn

$$\boxed{Mdx + Ndy = 0}$$

A differential eqn $Mdx + Ndy = 0$ is said to be an exact differential eqn if $\exists\, f(x,y)$ such that

$$d[f(x,y)] = Mdx + Ndy$$

e.g. $y^2\,dx + 2xy\,dy = 0$

$$d[xy^2] = y^2dx + 2xy\,dy$$

The D.E $Mdx + Ndy = 0$ is exact $\Longleftrightarrow$ $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

$Mdx + Ndy = 0$

$\downarrow$ exact

i.e. $\left(\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}\right)$

$\int M\,dx + \int (\text{terms of } N \text{ without } x)\,dy = c$

take y as constant

① $py\,dx + (1 + p^2 + \cos^2 x)\,dy = 0$ [illegible] exact eqn find p

$\Longrightarrow$ $\dfrac{\partial M}{\partial y} = p$ $\qquad \dfrac{\partial N}{\partial x} = -2\cos x \sin x$

$p = -2\cos x \sin x = $ [illegible]

$p\,dx - (1+\sin^2 y + \cos^2 x)\,dy = 0$ is exact then find p.

→ $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\therefore \frac{\partial N}{\partial x} = \sin 2x$

$\therefore \int \frac{\partial M}{\partial y} = \int \sin 2x$

$\therefore \boxed{P = y \sin 2x}$ ✓

$P = y\sin 2x + \cos x$ ✓ corr

3) $(3a^2x^2 + by\cos x)\,dx + (2\sin x - 4ay^3)\,dy = 0$ is exact then

(a) Exactness depends on both a and b

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

(b) Exactness depends only on a

(c) Exactness depends only on b

(d) Exactness not depends on both a & b.

④ $(ax + hy + g)\,dx + (hx + by + f)\,dy = 0$ Non homogeneous

$\rightarrow$ $\dfrac{\partial M}{\partial y} = h$, $\dfrac{\partial N}{\partial x} = h$

$\int (ax + hy + g)\,dx + \int (by + f)\,dy = c$

$\dfrac{ax^2}{2} + hxy + gx + \dfrac{by^2}{2} + fy = c.$

⑤ $[y(1 + 1/x) + \cos y]\,dx + [x + \log x - x\sin y]\,dy = 0$

$\rightarrow$ $\dfrac{\partial M}{\partial y} = -\sin y + \left(1 + \dfrac{1}{x}\right)$ $\dfrac{\partial N}{\partial x} = 1 + \dfrac{1}{x} - \sin y$

$\int \left(y + \dfrac{y}{x} + \cos y\right) dx + \int 0\,dx = c$

$\therefore$ $yx + y\log x + x\cos y = c$

⑥ $y\,dx - x\,dy = 0$ — ①

$\dfrac{\partial M}{\partial y} = 1$, $\dfrac{\partial N}{\partial x} = -1$

multiply ① with $1/y^2$

$\dfrac{y}{y^2}\,dx - \dfrac{x}{y^2}\,dy = 0$

$\dfrac{1}{y}\,dx - \dfrac{x}{y^2}\,dy = 0$

$$\frac{\partial M}{\partial y} = -\frac{1}{y^2}, \quad \frac{\partial N}{\partial x} = -\frac{1}{y^2}$$

Integrating factor -

A non exact eqn is converted to the exact by multiplying it with a function $f(x,y)$. Then $f(x,y)$ is called integrating factor.

$$y\,dx - x\,dy = 0 \xrightarrow{\text{I.F.}} \frac{1}{y^2}, \frac{1}{x^2}, \frac{1}{xy}, \frac{1}{x^2+y^2}$$

A constant multiple of an integrating factor is also an integrating factor.

$$\boxed{M\,dx + N\,dy = 0}$$

Non exact eqn (i.e. $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$)

i) All terms of M and N should be of same degree.

$$\boxed{IF = \frac{1}{Mx+Ny}} \quad (Mx+Ny \neq 0)$$

ii) The eqn is of the form

$$y\,f(xy)\,dx + x\,g(xy)\,dy = 0$$

$$\boxed{IF = \frac{1}{Mx-Ny}} \quad (Mx-Ny \neq 0)$$

iii) $\dfrac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N} = f(x)$ only or constant

$$\boxed{IF = e^{\int f(x)dx}}$$

iv) $\dfrac{\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}}{M} = g(y)$ only or constant

$$\boxed{IF = e^{\int g(y)dy}}$$

v) Inspection method

Note:

A non-exact homogeneous differential eqn $Mdx + Ndy$ is converted to exact by multiplying with $\left(\frac{1}{Mx+Ny}\right)$

Q. Find integrated factor of

$(y^4 + 2y)dx + (xy^3 + 2y^4 - 4x)dy = 0$

→

$\dfrac{\partial M}{\partial y} = 4y^3 + 2 \qquad \dfrac{\partial N}{\partial x} = y^3 - 4$

$\dfrac{\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}}{M} = \dfrac{y^3 - 4 - 4y^3 - 2}{y^4 + 2y} = \dfrac{-3(y^3+2)}{y(y^3+2)} = \dfrac{-3}{y}$

$IF = e^{\int -\frac{3}{y}dy} = e^{-3\log y} = y^{-3}$

$f(x)$	$e^{\int f(x)dx}$
$\frac{1}{x}$	x
$\frac{2}{x}$	x^2
$\frac{3}{x}$	x^3
$\frac{-1}{x}$	$x^{-1} = 1/x$
$-2/x$	$x^{-2} = 1/x^2$

① $y(1-xy)dx - x(1+xy)dy = 0$ — ①

→

$$Mx - Ny = xy - x^2y^2 + xy + x^2y^2 = 2xy$$

$$IF = \frac{1}{2xy} \quad \text{or} \quad IF = \frac{1}{xy}$$

Multiply ① with $1/xy$

$$\frac{y(1-xy)}{xy}dx - \frac{x(1+xy)}{xy}dy = 0$$

$$\int\left(\frac{1}{x} - y\right)dx - \int\frac{1}{y}dy = c$$

$$\log x - xy - \log y = c$$

$$\boxed{\log x/y - xy = c}$$

$$\frac{\partial M}{\partial y} \qquad \frac{\partial N}{\partial x}$$

$$I.F = 1/x^2$$

$$x^2 y\,dx - (x^3+y^3)\,dy = 0$$

$$\longrightarrow \quad Mx + Ny = x^3y - (x^3+y^3)y = -y^4$$

$$I.F = 1/y^4$$

$$\frac{x^2y}{-y^4}\,dx + \frac{x^3+y^3}{y^4}\,dy = 0$$

$$-\int \frac{x^2}{y^3}\,dx + \int \frac{1}{y}\,dy = c$$

$$-\frac{x^3}{3y^3} + \log y = c$$

* $y\,dx - x\,dy + (1+x)dx + x^2\cos y\,dy = 0$

multiply with $1/x^2$

$$\frac{y\,dx - x\,dy}{x^2} + \frac{(1+x)}{x^2}dx + \frac{x^2\cos y\,dy}{x^2} = 0$$

$$\int d\left[\frac{-y}{x}\right] + \int\left(\frac{1}{x^2}+\frac{1}{x}\right)dx + \int \cos y\,dy = 0$$

$$-\frac{y}{x} - \frac{1}{x} + \log x + \sin y = c$$

* $y\,dx - x\,dy + (y^2+y)x\,dx + xye^x\,dx = 0$

multiply with $1/xy$

$$\frac{y\,dx - x\,dy}{xy} + \frac{(y^2+y)x\,dx}{xy} + \frac{xy\,e^x\,dx}{xy} = 0$$

$$\int d\left[\log(x/y)\right] + \int (y+1)dy + \int e^x\,dx = 0$$

$$\log(x/y) + \frac{y^2+y}{2} + e^x = c$$

* $y\,dx + x\,dy + xe^{-x}dx = 0$

$$\int d(xy) + \int xe^{-x}dx = 0$$

$$xy - e^{-x}(x+1) = c$$

* $y\,dx + x\,dy + x^2 y\, e^x\, dx = 0$

multiply with $1/xy$.

$$\frac{y\,dx + x\,dy}{xy} + \frac{x^2 y\, e^x\, dx}{xy} = 0$$

$$\int d(\log(xy)) + \int x e^x\, dx = 0$$

$$\log(xy) + e^x(x-1) = c$$

* Linear differential eqns -

The differential eqn is said to be linear if it satisfies the following two conditions:

i) The dependent variable and all it's derivatives should be of 1st degree only i.e.

$(y')^1$, $\left(\frac{dy}{dx}\right)^1$, $\left(\frac{d^2y}{dx^2}\right)^1$,

ii) There is no product of the dependent variable and any of it's higher derivative.

$\left(y\frac{dy}{dx}\right)$, $\frac{dy}{dx}\frac{d^2y}{dx^2}$,

* $\frac{d^2y}{dx^2} + \frac{3dy}{dx} + xy = x^3$

$\Longrightarrow$ Linear

The eqn containing fn of dependent variable is not linear in that variable.

e.g.

$\frac{d^2y}{dx^2} + \frac{3dy}{dx} + xe^y = x^3$

$\Longrightarrow$ Non-linear

* Linear in y -

$$\boxed{\frac{dy}{dx} + Py = Q}$$

soln ↓ P and Q are fn of 'x' or constant.

$$\boxed{y \cdot e^{\int P dx} = \int Q e^{\int P dx} dx + c}$$

$$\boxed{IF = e^{\int P dx}}$$

* Linear in x -

$$\boxed{\frac{dx}{dy} + Px = Q}$$

soln ↓ P and Q are fn of 'y' or constant

$$x \cdot e^{\int P dy} = \int Q \cdot e^{\int P dy} dy + c$$

* $x \frac{dy}{dx} + y = x^4$ with $y(1) = 6/5$

→

$$\frac{dy}{dx} + \frac{y}{x} = x^3$$

$P = 1/x, \quad Q = x^3$

$$e^{\int \frac{1}{x} dx} = x$$

$$\therefore \; y \cdot x = \int x^3 \cdot x \, dx + c = \int x^4 dx + c$$

$$yx = \frac{x^5}{5} + c$$

at $y(1) = 6/5$

$$\frac{6}{5} = \frac{1}{5} + c$$

$$\boxed{c = 1}$$

$$\boxed{y = \frac{x^4}{5} + \frac{1}{x}}$$

* $x\cos x \frac{dy}{dx} + y(x\sin x + \cos x) = 1$

$\longrightarrow$

$$\frac{dy}{dx} + \left(\frac{x\sin x + \cos x}{x\cos x}\right)y = 1$$

$$\frac{dy}{dx} + (\tan x + 1/x)\,y = \frac{1}{x\cos x}$$

$$e^{\int p\,dx} = e^{\int(\tan x + 1/x)dx} = e^{\log\sec x + \log x}$$

$$= e^{\log(\sec x \cdot x)}$$

$$\therefore e^{\int p\,dx} = x\sec x.$$

$$y \cdot x\sec x = \int \frac{1}{x\cos x} \times x\sec x\,dx + c$$

$$\therefore xy\sec x = \tan x + c$$

* $x^2 \frac{dy}{dx} + 2xy = \frac{2\log x}{x}$ with $y(1) = 0$ then find y when $x = e$

$\longrightarrow$

$$\frac{dy}{dx} + \frac{2y}{x} = \frac{2\log x}{x^3}$$

$$e^{\int p\,dx} = e^{\int 2/x\,dx} = x^2$$

$$y \cdot x^2 = \int \frac{2\log x}{x^{\cancel{3}}} \times \cancel{x^2}\,dx + c$$

$\cancel{x}$ $\log x = t \Rightarrow \frac{1}{x}dx = dt$

$x^2y = \int 2t\,dt + c$

$x^2y = t^2 + c$

$x^2y = (\log x)^2 + c$

$\Rightarrow c = 0$

$x^2y = (\log x)^2$

when $x = e$ then

$e^2y = 1$

$\therefore \boxed{y = e^{-2}}$

* $(x + 2y^3)\frac{dy}{dx} = y$ with $x(1) = 0$

$\rightarrow$

$\cancel{\frac{x\,dy}{dx} + \frac{2y^3\,dy}{dx}} = y$ $\qquad \frac{x + 2y^3}{y} = \frac{dx}{dx}$

$\frac{dx}{dy} - \frac{x}{y} = 2y^2$

$e^{\int P\,dy} = e^{\int -1/y\,dy} = 1/y$

$$x \cdot \frac{1}{y} = \int 2y^2 \cdot \frac{1}{y} \, dy + c$$

$$\frac{x}{y} = y^2 + c$$

$x(1) = 0$

$\Longrightarrow$

$c = -1$

$$\frac{x}{y} = y^2 - 1$$

$$\boxed{x = y^3 - y}$$

* $\frac{dy}{dx} + y = e^x$ with $y(0) = 1$ find $y(1) = ?$

$\longrightarrow$

$$e^{\int p \, dx} = e^{\int 1 \, dx} = e^x$$

$$y \cdot e^x = \int e^x \cdot e^x \, dx + c$$

$$y e^x = \frac{e^{2x}}{2} + c$$

$y(0) = 1$

$\Rightarrow c = \frac{1}{2}$

$\therefore \; y e^x = \frac{e^{2x}}{2} + \frac{1}{2}$

At $y(1) =$

$\therefore \; y = \underline{\frac{e^x + e^{-x}}{2}}$ $\boxed{y = \frac{e^1 + e^{-1}}{2}}$

* Bernoulli's differential eqn :-

$$\boxed{\frac{dy}{dx} + Py = Q\,y^{n}} \rightarrow \text{Linear if } n=0.$$

P and Q are f^n of x (or) constants

sol^n

$$\boxed{y^{1-n}\, e^{\int(1-n)P\,dx} = \int (1-n)\,Q\, e^{\int(1-n)P\,dx}\, dx + c}$$

$$\boxed{y^{1-n} = v}$$ substitution

The substitution $y^{1-n} = v$ reduces Bernoulli's eqn to linear eqn.

GATE
Q. ... n $y^{1-n} = v$ reduces the non-linear eqn to which of the following linear form.

$\frac{d}{dx}$

$$\frac{dv}{dx} - \frac{1}{x}v$$

$$e^{\int \frac{1}{x}dx} = e^{\log x}$$

$$\frac{1}{1-n}\frac{dv}{dx} + Pv = Q$$

$$\boxed{\frac{dv}{dx} + (1-n)Pv = (1-n)Q}$$

Q. Which of the following substitution reduces the non-linear eqn $x\frac{dx}{dy} + x^2y^2 = x^3y^3$ to linear form?

(a) $y^{-1} = v$ (b) $y^{-2} = v$ ✓ (c) $x^{-1} = v$ (d) $x^{-2} = v$

→

$$\frac{dx}{dy} + xy^2 = x^2y^3$$

$$\therefore x^{1-2} = v$$

$$\therefore \boxed{x^{-1} = v}$$

Q. $\frac{dy}{dx} - y\tan x = -y^2\sec x$ with $y(0) = 1$

→

$$e^{\int (1-2)(-\tan x)dx} = e^{\log \sec x} = \sec x$$

$$y^{1-2}\sec x = \int (1-2)(-\sec x)(\sec x)\,dx + c$$

$$y^{-1}\sec x = \tan x + c$$

$$y(0) = 1 \implies c = 1$$

$$\boxed{y = \frac{\sec x}{1+\tan x}}$$

* Equations of the form :-

$$\boxed{f'(y)\frac{dy}{dx} + P f(y) = Q} \longrightarrow \text{Non-linear}$$

$$f(y) = v \implies f'(y)\frac{dy}{dx} = \frac{dv}{dx}$$

$$\boxed{\frac{dv}{dx} + Pv = Q} \longrightarrow \text{Linear}$$

* What substitution reduces the non-linear eqns $\frac{dz}{dx} + \frac{z}{x}\log z = \frac{z}{x^2}(\log z)^2$. Also find I.F. of linear form.

$\longrightarrow$

$$\frac{1}{z(\log z)^2}\frac{dz}{dx} + \frac{1}{x}(\log z)^{-1} = \frac{1}{x^2}$$

$$\boxed{(\log z)^{-1} = v}$$

$$-\frac{1}{z}(\log z)^{-2}\frac{dz}{dx} = \frac{dv}{dx}$$

$$-\frac{dv}{dx} + \frac{1}{x}v = \frac{1}{x^2}$$

$$\frac{dv}{dx} - \frac{1}{x}v = -\frac{1}{x^2}$$

$$e^{\int -\frac{1}{x}dx} = e^{\log x^{-1}} = 1/x$$

* $x\frac{dy}{dx} + y\log y = xye^{x}$

$\longrightarrow$

$$\frac{x}{y}\frac{dy}{dx} + \log y = xe^{x}$$

$$\log y = v$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{dv}{dx}$$

$$x\frac{dv}{dx} + v = xe^{x}$$

$$\frac{dv}{dx} + \frac{1}{x}v = e^{x}$$

$$e^{\int \frac{1}{x}dx} = x$$

$$v\cdot x = \int e^{x}\cdot x\,dx + c$$

$$vx = e^{x}\cdot(x-1) + c$$

$$x\log y = e^{x}(x-1) + c$$

* Clairaults eqn

$$\boxed{y = x\frac{dy}{dx} + f\left(\frac{dy}{dx}\right)}$$

or

$$\boxed{y = px + f(p)} \quad \text{where } p = \frac{dy}{dx}$$

soln

Directly replace $\frac{dy}{dx}$ by 'c'

$$\boxed{y = cx + f(c)}$$

* $\left(y - x\frac{dy}{dx}\right)\left(\frac{dy}{dx} - 1\right) = \frac{dy}{dx}$

→

$$y - xy' = \frac{y'}{y'-1}$$

$$y = \frac{y'}{y'-1} + xy'$$

$$\boxed{y = cx + \frac{c}{c-1}}$$

* $p = \sin(y - xp)$ where $p = \frac{dy}{dx}$

$\longrightarrow$

$$y - xp = \sin^{-1} p$$

$$y = xp + \sin^{-1} p$$

$$\boxed{y = cx + \sin^{-1} c}$$

* Higher order linear differential eqns with constant coefficients :-

$$\boxed{\frac{d^n y}{dx^n} + k_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + k_{n-1} \frac{dy}{dx} + k_n y = X}$$

$k_1, k_2, \cdots, k_n$ are constants

X - f^n of 'x' or constants.

$$\boxed{D = \frac{d}{dx}, \quad D^2 = \frac{d^2}{dx^2}, \quad D^3 = \frac{d^3}{dx^3}} \Longrightarrow \text{e.g. } De^{2x} = 2e^{2x}$$

$$\boxed{\frac{1}{D} = \int, \quad \frac{1}{D^2} = \iint \Longrightarrow \text{e.g. } \frac{1}{D}\cos 2x = \frac{\sin 2x}{2}}$$

$$(D^n + k_1 D^{n-1} + \cdots + k_{n-1} D + k_n) y = X$$

$$\boxed{F(D)\, y = X}$$

$F(D) = D^n + k_1 D^{n-1} + \cdots + k_{n-1} D + k_n$

~~f^n of differential~~ operator

(1) The complete solⁿ of $F(D)y = X$ is

~~Com~~ y = Complementary fⁿ + Particular integral
= CF + PI

(2) If $X = 0$ then $F(D)y = 0$ is called homogeneous linear differential eqn

(3) If $X \neq 0$ the $F(D)y = X$ is called non-homogeneous linear differential eqn

(4) The solution of homogeneous linear differential eqn $F(D)y = 0$ is called complementary fⁿ.

The no. of arbitrary constants in the complementary fⁿ should be equal to the order of given differential eqn.

(5) The particular integral of $F(D)y = X$ is

$$PI = \left[\frac{1}{F(D)}\right] X$$

. PI will not contain any arbitrary constants.

(6) If $X = 0$ then complete solⁿ of the given eqn is only complementary function.

(7) By assuming 'D' as an algebric ~~qan~~ quantity $F(D) = 0$ becomes an algebric eqn

And is called Auxillary eqn

- Procedure to find CF -

By solving an A.E. we get the roots based on nature of this roots we write the CF as follows:

Nature of roots	CF
Real and distinct $D = m_1, m_2, m_3$	$CF = C_1 e^{m_1 x} + C_2 e^{m_2 x} + C_3 e^{m_3 x}$
2 Real & repeated $D = m_1, m_1$	$CF = (C_1 + C_2 x) e^{m_1 x}$
③ Complex & distinct $D = a \pm ib$	$CF = e^{ax}[C_1 \cos bx + C_2 \sin bx]$
④ Complex & repeated $D = a \pm ib, a \pm ib$	$CF = e^{ax}[(C_1 + C_2 x)\cos bx + (C_3 + C_4 x)\sin bx]$
⑤ Surds (Real no.) $a \pm \sqrt{b}$	$CF = C_1 e^{(a+\sqrt{b})x} + C_2 e^{(a-\sqrt{b})x}$ OR $CF = e^{ax}[C_1 \cosh\sqrt{b}\,x + C_2 \sinh\sqrt{b}\,x]$

Roots | CF

$D = 1, -\frac{1}{2}, 2$ $\quad CF = c_1 e^{x} + c_2 e^{-x/2} + c_3 e^{2x}$

$D = 2, -2, -2$ $\quad CF = c_1 e^{2x} + (c_2 + c_3 x) e^{-2x}$

$D = 2 \pm 3i, \frac{1}{3}$ $\quad CF = e^{2x}[c_1 \cos 3x + c_2 \sin 3x] + c_3 e^{x/3}$

$D = \pm 2i, \pm 4$ $\quad CF = c_1 e^{4x} + c_2 e^{-4x} + [c_3 \cos 2x + c_4 \sin 2x]$

* $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$

$\rightarrow$

$D^2 - 5D + 6 = 0$

$\therefore D = +3, D = +2$

$CF = c_1 e^{+3x} + c_2 e^{+2x}$

* $\frac{d^4y}{dx^4} - 16y = 0$

$\rightarrow$

$(D^4 - 16)y = 0$

$D^4 - 16 = 0$

$(D^2 - 4)(D^2 + 4) = 0$

$D = 2, -2, \pm 2i$

$CF = c_1 e^{2x} + c_2 e^{-2x} + [c_3 \cos 2x + c_4 \sin$

* $\frac{d^2y}{dx^2} + 2p\frac{dy}{dx} + (p^2+q^2)y = 0$

$\longrightarrow$

$$D^2 + 2pD + (p^2+q^2) = 0$$

$$D_1, D_2 = \frac{-2p \pm \sqrt{4p^2 - 4p^2 - 4q^2}}{2} = \frac{-2p \pm \sqrt{-4q^2}}{2}$$

$$= \frac{-2p \pm 2qi}{2}$$

$$D_1, D_2 = -p \pm qi$$

$$CF = e^{-px} \cdot [c_1 \cos qx + c_2 \sin qx]$$

Note !

If $y = c_1y_1 + c_2y_2 + c_3y_3 + \cdots$ is the complete solution of homogeneous differential eqn $F(D)y=0$ then each one of $y_1, y_2, y_3, \cdots$ is linearly independent solution of the homogeneous of linear differential eqn $F(D)y=0$.

Que: y_1, y_2 are linearly independent solns of the corresponding ~~linearly~~ ~~indepe~~ homogeneous eqn of $F(D)y = X$ ~~then~~ $y_1 - y_2$ is a soln of which of the following eqns

(a) $F(D)=0$ (b) $X=0$

(c) $F(D)y = X$ ✓ (d) $F(D)y=0$

[illegible] $D = 3$ [illegible]

[illegible]

$y'' - 6y' + 8y = 0$

* $\sin 3x$, $\cos 3x$ are linearly independent [illegible]

$CF = c_1 \cos 3x + c_2 \sin 3x$

$D = \pm 3i$

$D^2 + 9 = 0$

$(D^2 + 9)y = 0$

$y'' + 9y = 0$

* e^{x}, e^{-x} are linearly indep[illegible]

[illegible]

[illegible] $y = 0$

* e^{3x}, e^{-3x} are linearly independent sol^n of -

(a) $\frac{d^2y}{dx^2} - \frac{6dy}{dx} + 9y = 0$ (b) $\frac{d^3y}{dx^3} + \frac{2d^2y}{dx^2} + \frac{3dy}{dx} + y = 0$

✓ (c) $\frac{d^4y}{dx^4} - 81y = 0$ (d) None

$$\boxed{\text{Order} \geq \text{No. of indep. sol}^{ns}}$$

* $y''' + y'' + y' + y = 0$

$\longrightarrow$

$D^3 + D^2 + D + 1 = 0$

$D^2(D+1) + (D+1) = 0 \quad \therefore (D^2+1)(D+1) = 0$

$\cancel{D^2=0}$, $D^2 = -1$, $D = -1$

$\therefore D = \pm i$

$y = c_1 e^{-x} + c_2 \cos x + c_3 \sin x$

* $\frac{d^4y}{dx^4} + a^4 y = 0$

$D^4 + a^4 = 0$

$(D^2 + a^2)^2 - 2D^2a^2 = 0$

$(D^2 + a^2)^2 - (\sqrt{2}Da)^2 = 0$

$(D^2+a^2+\sqrt{2}Da)(D^2+a^2-\sqrt{2}Da) = 0$

$D^2+\sqrt{2}aD+a^2=0$ & $D^2-\sqrt{2}aD+a^2=0$

$\frac{-\sqrt{2}a \pm \sqrt{2a^2-4a^2}}{2}$

$\frac{-\sqrt{2}a \pm \sqrt{-2a^2}}{2}$

$\frac{-\sqrt{2}a \pm \sqrt{2}ai}{2}$

$-\frac{a}{\sqrt{2}} \pm \frac{a}{\sqrt{2}}i$ $\qquad \frac{a}{\sqrt{2}} \pm \frac{a}{\sqrt{2}}i$

$$y = e^{-\frac{ax}{\sqrt{2}}}\left[c_1\cos\frac{a}{\sqrt{2}}x + c_2\sin\frac{a}{\sqrt{2}}x\right] + e^{\frac{ax}{\sqrt{2}}}\left[c_3\cos\frac{a}{\sqrt{2}}x + c_4\sin\frac{a}{\sqrt{2}}x\right]$$

* $\frac{d^2f}{dn^2} + 4\frac{df}{dn} + 4f = 0$

(a) $f_1 = e^{-2n}$, $f_2 = e^{2n}$ ✓(c) $f_1 = ne^{-2n}$, $f_2 = e^{-2n}$

(b) $f_1 = e^{-2n}$, $f_2 = ne^{2n}$ (d) $f_1 = ne^{-2n}$, $f_2 = e^{2n}$

* The D.E. $\frac{d^2y}{dx^2} + 2\cos x\frac{dy}{dx} - 3y = x^2$ is

(a) Homogeneous linear D.E. ✗(b) Non homogeneous linear eq with constant coeff

✗(c) Non linear D.E. order 2 ✓(d) Linear D.E.

The sol^n of

* $\frac{d^2 i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = 0$ where $R^2 C = 4L$

(a) $e^{Rt/2R}$ (b) $te^{Rt/L}$

(c) $e^{-Rt/L}$ ✓(d) $te^{-Rt/2L}$

→

$$D^2 + \frac{R}{L}D + \frac{1}{LC} = 0$$

$$D_1, D_2 = \frac{-\frac{R}{L} \pm \sqrt{\frac{R^2}{L^2} - \frac{4}{LC}}}{2}$$

$$= \frac{-\frac{R}{L} \pm \sqrt{\frac{R^2 C - 4L}{L^2 C}}}{2}$$

$$= \frac{-\frac{R}{L} \pm 0}{2}$$

$$D_1, D_2 = \frac{-R}{2L}, \frac{-R}{2L}$$

$$\therefore y = \cancel{CF}\,(C_1 + C_2 t)e^{-\frac{R}{2L}t}$$

$$\therefore y = C_1 e^{-Rt/2L} + C_2 t e^{-\frac{Rt}{2L}}$$

* $\frac{d^2y}{dx^2} + \frac{4dy}{dx} + 13y = 0$

with $y(0) = 0$ & $y'(0) = 1$

$\longrightarrow$

$D^2 + 4D + 13 = 0$

$D_1, D_2 = \frac{-4 \pm 6i}{2}$

$= -2 \pm 3i$

$y = e^{-2x}[c_1 \cos 3x + c_2 \sin 3x]$

at $x = 0$, $y = 0 \Rightarrow c_1 = 0$

$y = c_2 e^{-2x} \sin 3x$

$y' = c_2 [-2e^{-2x} \sin 3x + \cos 3x \, e^{-2x}]$

$y'(0) = 1$

$\therefore c_2 = 1/3$

$$\boxed{y = \frac{e^{-2x}}{3} \sin 3x}$$

* $9\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + y = 0$ with $y(0)=3$ & $y'(0)=1$

→

$$9D^2 - 6D + 1 = 0$$

$\therefore$ $D = 1/3, 1/3$

$$y = (C_1 + C_2 x)e^{x/3}$$

$y(0) = 3 \Rightarrow \boxed{3 = C_1}$

$\therefore$ $y = 3 + C_2 x e^{x/3}$

$$y' = C_2\left[\frac{x}{3}e^{x/3} + e^{x/3}\right]$$

$y'(0) = 1$

$\boxed{0 = C_2}$

$\therefore$ $\boxed{y = (3 + 0x)e^{x/3}}$

$\therefore$ $\boxed{y = 3e^{x/3}}$

* $\frac{d^2y}{dx^2}$ + ... with ...

(a) ... (b) ...

(c) $y = \sum_n c_n \sin\frac{n\pi x}{l}$ (d) $y = \sum_n ...$

* Direct put $cond^{ns}$ in options i.e. $y(0)=0$ *

* $\frac{d^2n}{dx^2} - \frac{n}{L^2} = 0$ where L is a constant & $n(0)=K$ & $n(\infty)=0$

(a) $n = Ke^{x/L}$ (b) $n = K^2 e^{-x/L}$

(c) $n = Ke^{-x/\sqrt{L}}$ (d) $n = Ke^{-x/L}$

Here check both $cond^n$ then c & d satisfies so from roots of A·E (d) is ans.

$D = \pm 1/L$

$y = c_1 e^{x/L} + c_2 e^{-x/L}$

* For what value of λ the D.E $\frac{d^2y}{dx^2} + \lambda y = 0$ with $y(0)=0$ and $y(\pi)=0$ will have non-trivial sol^n → Non-zero

(a) $\lambda = ...$ (b) ...

(c) $\lambda = 1, 3, 5$ (d) ...

$y(0)=0 \Rightarrow y = c_2 \sin\sqrt{\lambda}x$

$y(\pi)=0 \Rightarrow 0 = c_2 \sin ...$

$\sin\sqrt{\lambda}\pi = 0$ $= \sin n\pi$

$D = \pm\sqrt{\lambda}i$ $y = c_1\cos\sqrt{\lambda}x + c_2\sin\sqrt{\lambda}x$

$\sqrt{\lambda} = n$

$\lambda = n^2 = 1, 4, 9, ...$

Linked

1)* The complete soln of the differential eqn $\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = 0$ is $y = c_1e^{-x} + c_2e^{-3x}$ then values of p & q are –

(a) $p = 3,\ q = 3$ ✓(b) $p = 4,\ q = 3$

(c) $p = 4,\ q = 4$ (d) $p = 3,\ q = 4$

→

$(D+1)(D+3)y = 0$

$(D^2+4D+3)y = 0$

$D^2y + 4Dy + 3y = 0$

$p = 4,\ q = 3$

(2)* The soln of $\frac{d^2y}{dx^2} + p\frac{dy}{dx} + (q+1)y = 0$ is

→ (a) xe^{2x} (b) e^{2x} ✓(c) xe^{-2x} (d) x^2e^{-2x}

→

$y'' + 4y' + 4y = 0$

$D^2 + 4D + 4 = 0$

$(D+2)^2 = 0$

$D = -2, -2$

$y = (c_1 + c_2x)e^{-2x} = xe^{-2x}$ if $c_1 = 0,\ c_2 = 1$

Particular integrals :-

$$F(D)y = X$$

$$PI = \left[\frac{1}{F(D)}\right] X$$

- I → e^{ax} or constant
- II → $\sin ax$ or $\cos bx$
- III → x^m or poly. in x
- IV → $e^{ax} V$ → $\sin ax, \cos bx, x^m$, poly. in x

Case I - $X = e^{ax}$ or constant

$$PI = \left[\frac{1}{F(D)}\right] e^{ax} = \frac{1}{F(a)} e^{ax} \quad (\text{where } F(a) \neq 0)$$

Replace 'D' by 'a' in f^n $f(D)$

if $f(a) = 0$ then

$$PI = \cancel{x F(D)} \; x \left[\frac{1}{F'(D)}\right] e^{ax}$$

Replace 'D' by 'a' in $F'(D)$

$$PI = x \left[\frac{1}{F'(a)}\right] e^{ax} \quad (F'(a) \neq 0)$$

if $F'(a) = 0 \Rightarrow PI = x^2 \left(\frac{1}{F''(a)}\right) e^{ax} \quad (F''(a) \neq 0)$

* P.I. of $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 7y = e^{-2x}$

$\longrightarrow$

$$PI = \frac{1}{D^2+2D+7} e^{-2x} = \frac{1}{(-2)^2+2(-2)+7} e^{-2x}$$

$$\boxed{PI = \frac{e^{-2x}}{7}}$$

* PI of $\frac{d^2y}{dx^2} - 9y = e^{3x} + 3$

$\longrightarrow$

$$PI = \frac{1}{D^2-9}\left(e^{3x} + 3e^{0x}\right)$$

$$= x\left(\frac{1}{2D}\right)e^{3x} + 3\frac{1}{0-9}e^{0x}$$

$$= \frac{xe^{3x}}{6} + \frac{3}{-9}$$

$$\therefore PI = \frac{xe^{3x}}{6} - \frac{1}{3}$$

PI of $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = e^{-2x}$

$\rightarrow$

$$PI = \left[\frac{1}{D^2+4D+4}\right] e^{-2x} = x\left(\frac{1}{2D+4}\right) e^{-2x} = x^2\left(\frac{1}{2}\right)$$

$\therefore PI = \frac{x^2}{2} e^{-2x}$

* 'a' is the root of the A.E. of $F(D)y = e^{ax}$ then PI is (k constant)

(a) ke^{-ax} (b) ke^{ax} (c) kxe^{-ax} ✓(d) kxe^{ax}

* Case II -

$X = \sin ax$ (or) $\cos bx$ ($\sin(ax+b)$ or $\cos(cx+d)$)

$$PI = \left[\frac{1}{F(D)}\right] \sin ax = \left[\frac{1}{F(-a^2)}\right] \sin ax \quad (F(-a^2) \neq$$

Replace 'D^2' by '$-a^2$' in $F(D)$

if $F(-a^2) = 0$ -

$$PI = x\left[\frac{1}{F'(D)}\right] \sin ax = \frac{x}{F'(-a^2)} \sin ax$$

Replace 'D^2' by '$-a^2$' in $F'(D)$

* PI of $\frac{d^4y}{dx^4} + \frac{d^2y}{dx^2} + 3y = \cos 3x$

$\longrightarrow$

$$PI = \frac{1}{D^4+D^2+3} \cos 3x = \frac{1}{(-9)^2+(-9)+3} \cos 3x$$

$$PI = \frac{\cos 3x}{75}$$

* PI of $\frac{d^2y}{dx^2} + 4y = \cos(2x+4) + e^{-2x}$

$\longrightarrow$

$$PI = \frac{1}{D^2+4}\left(\cos(2x+4) + e^{-2x}\right)$$

$$= x\left(\frac{1}{2D}\right)\cos(2x+4) + \frac{1}{(-2)^2+4} e^{-2x}$$

$$= \frac{x}{2} \frac{\sin(2x+4)}{2} + \frac{e^{-2x}}{8}$$

$$\therefore PI = \frac{x \sin(2x+4)}{4} + \frac{e^{-2x}}{8}.$$

* PI of $\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = \sin x$

$\longrightarrow$

$$PI = \frac{1}{D^3+D^2+2D+2} \sin x$$

$$\frac{\sin x}{(-1)D + (-1) + 2D + 2} \qquad \frac{\sin x}{D + 1}$$

$$= \frac{\cos x - \sin x}{2}$$

$$PI = \frac{\sin x - \cos x}{2}$$

* case III $\quad X = x^m$ or poly. in x (m is +ve integer)

$$PI = \left[\frac{1}{F(D)}\right] x^m = \frac{1}{*[1 + \phi(D)]} x^m$$

$$= \frac{1}{*} [1 + \phi(D)]^{-1} x^m$$

* is taking least ~~power~~ common in $F(D)$

* PI of $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = x^2 + 1$

$\longrightarrow$

$$PI = \left(\frac{1}{D^2 + 2D + 2}\right)(x^2 + 1) = \frac{1}{2\left(1 + \frac{D^2 + 2D}{2}\right)} (x^2 + 1)$$

$$\frac{1}{2}\left[1 + \left(\frac{D^2 + 2D}{2}\right)\right]^{-1} (x^2 + 1)$$

30

$$= \frac{1}{2}\left[1-\left(\frac{D^2+2D}{2}\right)+\left(\frac{D^2+2D}{2}\right)^2 - \cdots\right](x^2+2)$$

$$= \frac{1}{2}\left[x^2+2-1-2x+2\right]$$

$$\therefore PI = \frac{x^2-2x+3}{2}$$

* PI of $\frac{d^3y}{dx^3} + \frac{3d^2y}{dx^2} = x^3+3x$

→

$$PI = \frac{1}{D^3+3D^2}(x^3+3x) = \frac{1}{3D^2\left[1+\frac{D^3}{3D^2}\right]}(x^3+3x)$$

$$= \frac{1}{3D^2}\left[1+\frac{D}{3}\right]^{-1}(x^3+3x)$$

$$= \frac{1}{3D^2}\left[1-\frac{D}{3}+\frac{D^2}{9}-D^3/27\cdots\right](x^3+3x)$$

$$= \frac{1}{3D^2}\left[x^3+3x-\frac{3x^2}{3}-1+\frac{6x}{9}-\frac{6}{27}\right]$$

$$= \frac{1}{3D^2}\left[x^3-x^2+\frac{33x}{9}-\frac{33}{27}\right] = \frac{1}{3D^2}\left[x^3-x^2+\frac{11}{3}x-\frac{11}{9}\right]$$

$$= \frac{1}{3}\left[\frac{x^5}{20}-\frac{x^4}{12}+\frac{11x^3}{18}-\frac{11x^2}{18}\right]$$

* Case IV :- $X = e^{ax} \cdot V$

$$PI = \left[\frac{1}{F(D)}\right]\left(e^{ax} \cdot V\right) = e^{ax}\left[\frac{1}{F(D+a)}\right]V$$

Replace 'D' by 'D+a' in F(D)

$$\boxed{\frac{1}{F(D+a)} V}$$

→ II if $V = \sin bx$ or $\cos bx$

→ III if $V = x^m$ or poly. in x